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The general projective group occupies a position of special importance in Lie's theory of finite continuous groups. For associated with any finite continuous group G_r with r parameters is a sub-group with $\rho \leqq r-1$ parameters of the general projective group in $(r-1)$-fold space, the knowledge of whose invariants (general and special) enables us to enumerate the different types of sub-groups of G_r. This projective group is obtained from the adjoined group of G_r by regarding the variables in the equations of transformation of the adjoined as homogeneous co-ordinates.

Lie showed that the general projective group is continuous, in the sense that each transformation of this group can be generated by an infinitesimal transformation of the group.* But Professor Study made the important discovery that not every transformation of the special linear homogeneous group can be generated by an infinitesimal transformation of the special linear homogeneous group; † and thus showed that the sub-groups of the general projective group are not all continuous in the sense in which this term is here employed: namely, a group is here termed continuous if each transformation of the group can be generated by an infinitesimal transformation of the group, and therefore belongs to a continuous one-term sub-group of the group in question.

Subsequently Professor Taber showed that not every transformation of the orthogonal group in n variables, for $n \geqq 4$, can be generated by an infinitesimal transformation of this group, and established equivalent results for the group of automorphic linear transformations of an alternate bilinear form, and for the group of automorphic linear transformations of a general bilinear form.‡ I have found, moreover, that a number of the

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* Lie, *Continuierliche Gruppen*, p. 45.

† *Leipzige Berichte*, 1892.

‡ *Bull. N. Y. Math. Society*, July, 1894; *Math. Ann.*, Vol. XLVI. p. 561; *Math. Review*, Vol. I. p. 154.

sub-groups of the projective group in two and three variables are not properly continuous, except in the neighborhood of the identical transformation. These groups are enumerated at the end of this paper.

In what follows I deal with a consequence of Study's discovery which I believe has not yet been touched upon. I shall term a transformation of a so called continuous group that cannot be generated by an infinitesimal transformation of this group a *singular transformation* of this group.

Let G_ρ denote a projective group in n-fold space. Two points, p and p_1, of general position on the same invariant manifold relative to G_ρ can always be interchanged by one or more transformations of G_ρ. In general, each of the transformations by which p and p_1 are interchanged can be generated by an infinitesimal transformation of G_ρ: in which case I shall say that the points p and p_1 can be continuously interchanged by the transformations of this group. But, if G_ρ contains singular transformations, it sometimes happens that the points p and p_1 cannot be interchanged by a transformation of G_ρ that can be generated by an infinitesimal transformation of G_ρ; and, in this case, I shall say that the points p and p_1 cannot be continuously interchanged.

If now $n = r - 1$, and G_ρ is the projective group above referred to, associated with an r-term group G_r, then every point in the space S_{r-1} to which the transformations of G_ρ are applied represents a one-term group of G_r. And two points, p and p_1, of general position on the same invariant manifold in S_{r-1}, relative to G_ρ, represent one-term groups of G_r of the same type, since they can be interchanged by transformations of G_ρ. If, however, p and p_1 cannot be continuously interchanged by the transformations of G_ρ, i. e. interchanged by a transformation generated by an infinitesimal transformation of G_ρ, the one-term groups of G_r represented by these points, although of the same type, are differently related from two one-term groups of G_r represented by two points of S_{r-1} that can be interchanged continuously by the transformations of G_ρ, i. e. interchanged by a transformation generated by an infinitesimal transformation of G_ρ.

If the smallest invariant manifold relative to any ρ-term projective group G_ρ is q-way extended, $q \leqq \rho$, then there are $\infty^{\rho - q}$ transformations of G_ρ that will interchange two points, p and p_1, of general position on any invariant manifold relative to G_ρ. If $\rho = q$, then there is but *one* transformation.* If this transformation is singular, that is, if this trans-

* Lie, *Continuierliche Gruppen*, p. 432.

formation cannot be generated by an infinitesimal transformation of G_ρ, then, clearly, not all points on each smallest invariant manifold can be continuously interchanged ; and, therefore, the one-term sub-groups of G_r represented by these points, if G_ρ is the projective group associated with G_r, cannot all be transformed into one another continuously by means of the transformation of G_ρ. But, if $q < \rho$, then it is by no means certain, when G_ρ contains singular transformations, that p and p_1 can be chosen so that all the $\infty^{\rho-q}$ transformations are singular. In fact, in all cases I have considered this is never possible. It may happen that but one or all but one of the $\infty^{\rho-q}$ transformations are singular. In this case the points of general position on any smallest invariant manifold can be continuously interchanged by means of the transformation of the given group, although the group contain transformations that cannot be generated by an infinitesimal transformation of the group.

I have examined all the two and three-term groups enumerated by Lie in the *Continuierliche Gruppen*, pp. 288 and 519. In each case the associated (adjoined) projective group G_ρ is such that two points of general position on the smallest invariant manifold relative to G_ρ can always be interchanged continuously, notwithstanding that in certain cases the associated group G_ρ contains singular transformations. I have therefore, as yet, found no group G_r whose one-term sub-groups of the same type cannot all be continuously interchanged by the transformations of the adjoined projective group. But it seems probable that such groups G_r exist.

The following examples illustrate the effect of the existence of singular transformations among the transformations of a projective group G_ρ upon the interchange, by transformations of G_ρ, of points on the same invariant manifold relative to G_ρ. They have been selected from the list given at the end of this paper. The third group considered is the adjoined group of a number of three-term projective groups.

Example 1. Consider the two-term projection group of the plane,

$$xq, \quad xp + 3yq.$$

The symbol of infinitesimal transformation is

$$c_1 xq + c_2 (xp + 3yq);$$

and the ∞^2 of finite transformations T_c generated by ∞^1 of infinitesimal transformations are of the form,

$$x' = e^{c_2} x_3,$$

$$y' = \frac{c_1}{2 c_2} (e^{3c_2} - e^{c_1}) x + e^{3c_2} y.$$

The group contains singular transformations T which are of the form,

$$x' = - x,$$
$$y' = Nx - y \quad (N \ddagger 0).$$

Now T applied to a point p on the line $x = + c$ will transform p to a point p_1 on the line $x = - c$; and, clearly, there is no non-singular transformation T_c among the transformations of the group that has the same effect. If the singular transformation T is applied to a point on the special invariant $x = 0$, p will be conveyed across the invariant point $x = 0$, $y = 0$. But this can be done by a non-singular transformation whose path curve is imaginary; for this transformation may be effected by the non-singular transformation

$$x' = - x,$$
$$y' = - y.$$

Therefore, two points, p and p_1, in the plane that lie on opposite sides of, and equidistant from, the special invariant $x = 0$ cannot *always* be interchanged among themselves continuously by means of the tranformations of the group.

Example 2.

$$x_3 p_2, \quad x_1 p_2, \quad x_1 p_1 + 2 x_2 p_2.$$

The ∞^3 of non-singular transformations T_c have the form,

$$x'_1 = e^{c_3} x_1,$$
$$x'_2 = \frac{c_2}{c_3} (e^{2c_3} - e^{c_3}) x_1 + e^{2c_3} x_2 + \frac{c_1}{2 c^3_3} (e^{2c_3} - 1) x_3.$$
$$x'_3 = x_3;$$

and the ∞^2 of singular transformation T have the form,

$$x'_1 = - x_1,$$
$$x'_2 = M x_1 + x_2 + N x_3, \quad (N \pm 0)$$
$$x'_3 = x_3.$$

A ∞^1 of the singular transformations T will move a given point p of general position on the line $x_1 = +c$, $x_3 = k$, to a given point p_1 of general position on the line $x_1 = -c$, $x_3 = k$. Nevertheless, we can find one non-singular transformation that will do the same, namely,

$$\begin{aligned} x'_1 &= -x_1, \\ x'_2 &= A x_1 + x_2, \\ x'_3 &= x_3. \end{aligned}$$

For clearly, by a proper choice of A, this transformation T_c has the same effect when applied to a definite given point as the transformation T for any given values of M and N $(N \neq 0)$.

Example 3.

$$x_3 p_1, \quad x_3 p_2, \quad x_1 p_1 + 2 x_2 p_2.$$

The ∞^3 of non-singular transformations T_c have the form,

$$\begin{aligned} x'_1 &= e^{c_3} x_1 + \frac{c_1}{c_3}(e^{c_3} - 1) x_3, \\ x'_2 &= e^{2c_3} x_2 + \frac{c_2}{2c_3}(e^{2c_3} - 1) x_3, \\ x'_3 &= x_3. \end{aligned}$$

The ∞^2 of singular transformations have the form,

$$\begin{aligned} x'_1 &= -x_1 + M x_3, \\ x'_2 &= x_2 + N x_3, \quad (N \neq 0) \\ x'_3 &= x_3. \end{aligned}$$

By means of the latter a given point p of general position on the plane $x_3 = k$ can be transformed into a given point p_1 of general position in that plane. But it is easily seen that c_1, c_2, and c_3 of T_c can be chosen in ∞^1 of ways so that T_c will produce the same effect.

Example 4.

$$x_3 p_1, \quad x_3 p_2, \quad 2 x_1 p_2 + 3 x_2 p_2 + x_3 p_3.$$

The ∞^3 of non-singular transformations T_c have the form,

$$x'_1 = e^{2c_3} x_1 + \frac{c_1}{c_3}(e^{2c_3} - e^{c_3}) x_3,$$

$$x'_2 = e^{3c_3} x_2 + \frac{c_2}{2\,c_3}\left(e^{3c_3} - e^{c_3}\right) x_3 ,$$
$$x'_3 = e^{c_3} x_3 .$$

The ∞^2 of singular transformations T have the form,

$$x'_1 = x_1 + M x_3 ,$$
$$x'_2 = -\, x_2 + N x_3 , \quad (N \pm 0)$$
$$x'_3 = -\, x_3 .$$

The transformation T, if we regard x_1, x_2, x_3 as Cartesian co-ordinates, will convey a given point p of general position in the plane $x_3 = + c$ to a point p_1 on the plane $x_3 = - c$; and clearly there is no other transformation of the group that will do the same. The points on the special invariant $x_3 = 0$ can be continuously interchanged, for the transformation effected by T can also be effected by the non-singular transformation,

$$x'_1 = x_1 + A x_3 ,$$
$$x'_2 = -\, x_2 ,$$
$$x'_3 = -\, x_3 .$$

Therefore, in this group, points on opposite sides of, and equidistant from, the special invariant $x_3 = 0$ cannot *all* be interchanged continuously among themselves.

The following groups enumerated by Lie on pp. 288 and 519 of his *Continuierliche Gruppen* are not properly continuous except in the neighborhood of the identical transformation.

$q, \; p + xq, \; xp + 2yq$	$q, \; xq, \; p + yq$
$p, \; q, \; xp + (y - x) q$	$xq, \; xp - yq, \; yp$
$p, \; q, \; (a - 1) xp + ayq$	$q, \; xq, \; xp + ayq$
$q, \; yq + p$	$xq, \; xp + q$

$$\boxed{xq, \; xp + ayq \; (a \pm 0, \; 1)}$$

$$\boxed{x_3p_2, \quad x_3p_1 + x_1p_2, \quad x_2p_2 - x_3p_3 + \beta U}$$

$$\boxed{x_3p_2, \quad x_1p_2, \quad x_3p_1 + x_2p_2 + \beta U}$$

$$\boxed{x_3p_1, \quad x_3p_2, \quad x_1p_1 + x_1p_2 + x_2p_2 + \beta U}$$

$$\boxed{x_1p_2, \quad x_1p_1 - x_2p_2, \quad x_2p_1}$$

$$\boxed{x_3p_1, \quad x_3p_2, \quad \alpha x_1p_1 + \beta x_2p_2 + \gamma x_3p_3}$$

$$\boxed{x_3p_2, \quad x_1p_2, \quad \alpha x_1p_1 + \beta x_2p_2 + \gamma x_3p_3}$$

$$\boxed{x_3p_2, \quad x_3p_1 + x_2p_2, \quad U}$$

$$\boxed{x_1p_2, \quad x_1p_1 + x_3p_2, \quad U}$$

$$\boxed{x_3p_2, \quad \alpha x_1p_1 + \beta x_2p_2, \quad U}$$

$$\boxed{x_3p_2, \quad x_3p_1 + x_2p_2 + \beta U}$$

$$\boxed{x_1p_2 + x_1p_1 + x_3p_2 + \beta U}$$

$$\boxed{x_3p_2, \quad \alpha x_1p_1 + \beta x_2p_2 + \gamma x_3p_3}$$